JOURNAL OF COMBINATORIAL THEORY, Series A 51, 90–103 (1989)

Anticlusters and Intersecting Families of Subsets

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Communicated by the Managing Editors

Received March 16, 1987

Given $\mathbf{B} \subseteq 2^{[n]}$, where $[n] = \{1, ..., n\}$, let $v(\mathbf{B})$ denote the maximum size of a family $\mathbf{F} \subseteq 2^{[n]}$ such that the intersection of any two sets in $\mathbf{F}$ contains some set in $\mathbf{B}$. Chung, Frankl, Graham, and Shearer conjecture that for any $X \subseteq [n]$, if $\mathbf{B}$ consists of all cyclic translates of X modulo n, then $v(\mathbf{B}) = 2^{n-|X|}$. Here we solve the weaker problem of proving that $v(\mathbf{B}) = 2^{n-|X|}$ when $\mathbf{B}$ consists of all translates of X in $[n]$ (not cyclic). We also show for all X that the conjecture of Chung *et al.* holds for infinitely many values of n. Our approach is to use linear algebra over $\mathbf{Z}_2$ to partition $2^{[n]}$ into $2^{n-|X|}$ families, called anticlusters, with the property that no set in $\mathbf{B}$ belongs to the intersection of any two sets in the same anticluster.

1. INTRODUCTION

Let $[n]$ denote the set $\{1, 2, ..., n\}$, and let $2^{[n]}$ denote the collection of all subsets of $[n]$. Given $\mathbf{B} \subseteq 2^{[n]}$, we say that $\mathbf{F} \subseteq 2^{[n]}$ is an *intersecting family over* $\mathbf{B}$ if for every F, F' in $\mathbf{F}$ there exists B in $\mathbf{B}$ such that $B \subseteq F \cap F'$. Following the notation of [1], the size of the largest intersecting family $\mathbf{F} \subseteq 2^{[n]}$ over $\mathbf{B}$ is denoted by $v(\mathbf{B})$.

If $\mathbf{B}$ consists of a single set $B_0 \subseteq [n]$ of size t, then there is a unique largest intersecting family over $\mathbf{B}$, which consists of all the subsets that contain B_0. Such a system is called a *kernel system* with kernel B_0. So in this case, $v(\mathbf{B}) = 2^{n-t}$. At the other extreme, suppose that $\mathbf{B} = \binom{[n]}{t}$, the collection of *all* t-subsets of $[n]$. That is, we require that $|F \cap F'| \geqslant t$ for all F, F' in $\mathbf{F}$. Katona [5] solved this problem in 1964:

$$v\left(\binom{[n]}{t}\right) = \begin{cases} \sum_{j \geqslant p} \binom{n}{j} & \text{if } n+t=2p \\ \sum_{j \geqslant p} \binom{n}{j} + \binom{n-1}{p-1} & \text{if } n+t=2p-1. \end{cases} \tag{1}$$

In this paper we consider some intermediate cases where $\mathbf{B} \subseteq \binom{[n]}{t}$ contains more than one set, yet $v(\mathbf{B})$ is still 2^{n-t}, so that a kernel system is still extremal.

* Research supported in part by the National Science Foundation under Grant DMS-8401281.

0097-3165/89 $3.00

Suppose that $X \in \binom{[n]}{t}$ with $X = \{x_1 < x_2 < \cdots < x_t\}$. For $1 - x_1 \leqslant i \leqslant n - x_t$, the translate $X + i = \{x_1 + i, ..., x_t + i\}$ is contained in $[n]$. The collection of such translates of X is denoted by $\mathbf{B}_n^*(X)$. Let $\mathbf{B}_n(X)$ denote the collection of all cyclic translates of X in $[n]$, that is, the sets $X + i$ where addition is carried out modulo n. Then we have immediately that

$$2^{n-t} \leqslant v(\mathbf{B}_n^*(X)) \leqslant v(\mathbf{B}_n(X)). \tag{2}$$

R. Graham [4] has offered \$100 US for a proof of the following conjecture.

CONJECTURE 1. *For all X in $\binom{[n]}{t}$, $v(\mathbf{B}_n(X)) = 2^{n-t}$.*

This conjecture appears in the paper of Chung, Frankl, Graham, and Shearer [1], where various results in support of the conjecture are proven. In this paper we propose a new approach to the problem. This approach may lead to a proof of the conjecture, although unfortunately (for us!) it is not yet clear how to do this. However, the method yields some new results, most notably that $v(\mathbf{B}_n^*(X)) = 2^{n-t}$. It also gives the simplest proof of the known result that the conjecture holds when X consists of t consecutive numbers.

2. ANTICLUSTERS

Conjecture 1 holds trivially for $t = 0$ and $t = n$. The next easiest case is $t = 1$: when $|X| = 1$, $\mathbf{B}_n(X) = \binom{[n]}{t}$; so we need to show that the largest size of a family of subsets, no two being disjoint, is 2^{n-1}. This is directly proven by observing that such a family $\mathbf{F}$ cannot contain both a set and its complement, so that $\mathbf{F}$ contains at most half of the 2^n sets in $2^{[n]}$. In this case, the inequality $v(\mathbf{B}_n(X)) \leqslant 2^{n-1}$ is induced by a partition of $2^{[n]}$ into 2^{n-1} blocks where no two sets in a block intersect.

This motivates our idea for general X. Call a collection $\mathbf{A} \subseteq 2^{[n]}$ an *anticluster* for $\mathbf{B} \subseteq 2^{[n]}$ if for every A, A' in $\mathbf{A}$ such that $A \neq A'$, the set $A \cap A'$ includes no set B in $\mathbf{B}$. Then for any intersecting family $\mathbf{F}$ (which we could call a *cluster*) and any anticluster $\mathbf{A}$ for $\mathbf{B}$, $|\mathbf{F} \cap \mathbf{A}| \leqslant 1$. It follows that for any $\mathbf{B}$, $v(\mathbf{B})$ is at most the minimum number of anticlusters needed to cover (equivalently, partition) $2^{[n]}$. Combining this observation with (2), we see that Conjecture 1 is implied by

CONJECTURE 2. *For all X there exists a partition of $2^{[n]}$ into 2^{n-t} anticlusters for $\mathbf{B}_n(X)$.*

In contrast to Conjecture 2 we offer an example which shows that for general $\mathbf{B}$, $v(\mathbf{B})$ anticlusters are not sufficient to cover $2^{[n]}$.

EXAMPLE. Let $n=6$ and $\mathbf{B}=\binom{[6]}{2}$, so that by (1), $v(\mathbf{B})=\binom{6}{6}+\binom{6}{5}+\binom{6}{4}=22$. Suppose there were a partition into just 22 anticlusters. The 22 subsets of size 4 or more intersect at least twice, so belong to distinct anticlusters. No 3-subset belongs to an anticluster containing a 5-subset or a 6-subset, and at most one 3-subset belongs to an anticluster with a 4-subset. So at least $\binom{6}{3}-\binom{6}{4}=5$ 3-subsets are left unaccounted for, a contradiction. In fact it can be shown that 24 anticlusters are necessary and sufficient.

The general covering problem, then, is to determine how many anticlusters for $\mathbf{B}$ are necessary to cover $2^{[n]}$.

3. A Matrix Conjecture

We now identify each subset $B \subseteq [n]$ with its characteristic vector $(a_1, ..., a_n)$, where $a_i = 1$ if $i \in B$ and $a_i = 0$ otherwise, viewed as an element of the vector space $\mathbf{Z}_2^n$ of dimension n over $GF(2)$. We seek to partition $\mathbf{Z}_2^n$ into 2^{n-t} affine subspaces of size 2^t, each parallel to a particular t-dimensional subspace, such that each corresponds to an anticluster for $\mathbf{B}(X)$. Of course any t-dimensional subspace $\mathbf{T}$ yields a partition of $\mathbf{Z}_2^n$ into 2^{n-t} affine subspaces of the form $\mathbf{v}+\mathbf{T}$, where $\mathbf{v} \in \mathbf{Z}_2^n$. The affine subspace $\mathbf{v}+\mathbf{T}$ corresponds to an anticluster for a collection $\mathbf{B} \subseteq \binom{[n]}{t}$ if and only if for all B in $\mathbf{B}$, no two elements of $\mathbf{v}+\mathbf{T}$ both have ones in every component indexed by the set B. Since $\mathbf{v}$ ranges over all elements of $\mathbf{Z}_2^n$, we see that the affine subspaces parallel to $\mathbf{T}$ all correspond to anticlusters for $\mathbf{B}$ if and only if for all B in $\mathbf{B}$, no two elements agree in every component indexed by B. That is, for all B, the projection of $\mathbf{T}$ into the t-dimensional subspace of $\mathbf{Z}_2^n$ generated by the standard basis vectors $\mathbf{e}_j$, $j \in B$, must be one to one and onto.

By considering a $t \times n$ matrix M which has for its rows the basis vectors for $\mathbf{T}$, this means that for all B the rows of the $t \times t$ submatrix M_B consisting of columns indexed by B span $\mathbf{Z}_2^t$, i.e., M_B is nonsingular. Thus Conjecture 2 is implied by the following conjecture about matrices.

CONJECTURE 3. *For all t and n, $0 \leqslant t \leqslant n$, and all t-subsets X of $[n]$, there exists a $t \times n$ matrix M such that for all i, the t columns of M indexed by the cyclic translate $X+i \pmod n$ are linearly independent over $GF(2)$.*

Another way of stating the condition above which may prove helpful is to require that each submatrix M_B has a determinant over the integers that is congruent to 1 modulo 2, i.e., each M_B has an odd number of transversals consisting entirely of ones. Another version of Conjecture 3 is given later in Section 8.

EXAMPLE. For $X=\{1\}$, so that $\mathbf{B}_n(X)=\binom{[n]}{1}$, simply take $M=[1\ 1\ \cdots\ 1]$. Each anticluster then consists of a set and its complement, the same partition we saw earlier.

EXAMPLE. For $X=\{1, 2\}$, an M that obviously works is

$$M=\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \end{bmatrix}. \tag{3}$$

One only has to observe that pairs of circularly consecutive columns are linearly independent. Next consider an arbitrary 2-subset X, say $X=\{i, i+d\}$. Let $g=\gcd(d, n)$. Walking around a cycle of length n in steps of size d decomposes the cycle into g cycles of length n/g. Using this idea, for column number $j+(r-1)\,d \pmod n$, when $1\leqslant j\leqslant g$ and $1\leqslant r\leqslant n/g$, use column r of (3) above. For example, if $X=\{3, 7\}$ and $n=10$ we take

$$M=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

So Conjecture 3 holds for $t\leqslant 2$.

An equivalent way of viewing the space $\mathbf{Z}_2^n$ that can be stated purely in terms of sets is used in [1]. We may view the space $\mathbf{Z}_2^n$ as containing the subsets of $[n]$, in which the group operation (addition in $\mathbf{Z}_2^n$) corresponds to the symmetric difference operation $\triangle$ on the subsets. Following [1], for sets F, $F'\subseteq[n]$, denote by $F \bigtriangledown F'$ the complement of the symmetric difference of F and F':

$$F \bigtriangledown F'=\overline{F \triangle F'}=(F\cap F')\cup(\bar{F}\cap \bar{F}').$$

Given $\mathbf{B}\subseteq 2^{[n]}$ we let $\bar{v}(\mathbf{B})$ denote the maximum size of a family $F\subseteq 2^{[n]}$ such that for every F, F' in $\mathbf{F}$, $F \bigtriangledown F'$ contains some member B of $\mathbf{B}$. Since $F \bigtriangledown F'\supseteq F\cap F'$, it follows that $v(\mathbf{B})\leqslant\bar{v}(\mathbf{B})$.

Now consider any $\mathbf{B}\subseteq\binom{[n]}{t}$ such that there exists a matrix M with the desired property. Then there exists $\mathbf{T}$ such that no two vectors in any translate $\mathbf{v}+\mathbf{T}$ of $\mathbf{T}$ agree on every component indexed by any B in $\mathbf{B}$. Thus for no two corresponding sets F, F' does $F \bigtriangledown F'$ contain any member B of $\mathbf{B}$. This means that the blocks of the partition of $2^{[n]}$ induced by M are not merely anticlusters for the operation $\cap$, but also for the operation $\bigtriangledown$. Thus, if true, Conjecture 3 would prove the stronger result that

$$\bar{v}(\mathbf{B}_n(X))=2^{n-t} \qquad \text{for all} \quad X\in\binom{[n]}{t}.$$

However, this was shown to follow from the weaker Conjecture 1 in view of the observation in [1] that

$$v(\mathbf{B}) = \bar{v}(\mathbf{B}) \qquad \text{for all} \quad \mathbf{B} \subseteq 2^{[n]}.$$

4. Blocks of Consecutive Integers

We now prove Conjecture 3 in the case where X is a block of t consecutive integers modulo n by explicitly constructing the required matrix M. This implies Conjecture 1 for such X, a result due to Chung, Frankl, Graham, and Shearer [1] that was also proven by Faudree, Schelp, and Sós [3, cf. 2]:

Theorem 1. *$v(\mathbf{B}_n([t])) = 2^{n-t}$ for all $0 \leqslant t \leqslant n$.*

Proof. The linear independence of columns of M is not affected by a change of basis for the column space, $\mathbf{Z}_2^t$, so for convenience we take the first t columns to be the identity matrix. For the remaining $n-t$ columns, we use Pascal's triangle (modulo 2) on its side. That is, we take $M = [m_{i,j}]$, where

$$m_{i,j} = \begin{cases} \delta_{i,j}, & 1 \leqslant i, j \leqslant t \\ \binom{j-(i+1)}{j-(t+1)}, & 1 \leqslant i \leqslant t, \quad t+1 \leqslant j \leqslant n. \end{cases} \tag{4}$$

For computational purposes, Pascal's recursion

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

yields this simpler description of the last $n-t$ columns of M:

$$\begin{cases} m_{i,t+1} = 1 & \text{for all} \quad i, \\ m_{t,j} = 1 & \text{for all} \quad j \geqslant t+1, \\ m_{i,j} = m_{i+1,j} + m_{i,j-i} \pmod 2 & \text{for} \quad 1 \leqslant i \leqslant t-1, \quad t+2 \leqslant j \leqslant n. \end{cases}$$

Example. For $t = 3$ and $n = 7$ this produces

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

It must be shown that any t consecutive columns of M are linearly independent over $GF(2)$, or equivalently, that they form a matrix with determinant $\equiv 1 \pmod 2$. Let $k \geqslant 0$ be the number of these t columns that are among the first t columns of M, and expand along those columns. If $k=t$, the determinant is trivially 1. For $k<t$, this reduces the problem to that of computing the determinant for a square block in the modulo 2 Pascal's triangle, the block being of order $t-k$ and touching the all ones border (either at column $t+1$ or at row t of M). It remains to prove that such determinants are all $\equiv 1 \pmod 2$.

Notation. Given a matrix A, let $A[a \cdots b,\ c \cdots d]$ denote the submatrix formed by selecting the rows numbered a through b and columns numbered c through d.

LEMMA. *If A is the infinite matrix with rows and columns indexed by natural numbers and entries defined by $A[i,j] = \binom{i+j}{i}$, then every submatrix of the form $A[0 \cdots m,\ r \cdots r+m]$ or $A[r \cdots r+m,\ 0 \cdots m]$ has determinant* 1.

Proof of Lemma. By symmetry of A, it is enough to consider the first case, $A[0 \cdots m,\ r \cdots r+m]$. We will use induction on m and r. If $m=0$, then the result is trivially true. Let us assume that $m>0$. Perform the following sequence of row operations on $A[0 \cdots m,\ r \cdots r+m]$:

Subtract row $m-1$ from row m,

subtract row $m-2$ from row $m-1$,

$\vdots$

subtract row 0 from row 1.

Call the result $B[0 \cdots m,\ r \cdots r+m]$. By Pascal's recursion on binomial coefficients, $A[i,j] - A[i-1,j] = A[i,j-1]$ when $j>0$. Therefore, if $r>0$ then $B[0 \cdots m,\ r \cdots r+m] = A[0 \cdots m,\ r-1 \cdots r-1+m]$. Thus in that case, $A[0 \cdots m,\ r \cdots r+m]$ has the same determinant as $A[0 \cdots m,\ r-1 \cdots r-1+m]$. By induction on r, it remains to consider the case $r=0$. When $r=0$, the sequence of row operations still has the effect of shifting the submatrix to the left, except that the leftmost column "falls of the edge." The first column of $B[0 \cdots m,\ 0 \cdots m]$ is $[1, 0, 0, ..., 0]^{\mathrm{T}}$. Expanding the determinant along that column shows that

$$\begin{aligned}\det A[0 \cdots m, 0 \cdots m] &= \det B[1 \cdots m, 1 \cdots m]\\ &= \det A[1 \cdots m, 0 \cdots m-1].\end{aligned}$$

By symmetry of A,

$$\det A[1 \cdots m, 0 \cdots m-1] = \det A[0 \cdots m-1, 1 \cdots m].$$

By inductive hypothesis on m, the latter determinant is 1, so we are done. ∎

This completes the proof of the theorem. ∎

In [1] a slightly stronger result is obtained: The family $\mathbf{B}_n([t])$ is enlarged to include the cyclic translates modulo $n-1$, $\mathbf{B}_{n-1}([t])$, and the operation $\cap$ is replaced by the operation ∇. We have already seen that whenever M can be constructed we can replace $\cap$ by ∇. The fact that for $X=[t]$ and for all m, n, the matrix M_m for $\mathbf{B}_m(X)$ forms the first m columns of the matrix M_n for $\mathbf{B}_n(X)$ implies that the columns of M_n selected by any translate of X modulo m are linearly independent. We combine these observations and record this strengthening of the result in [1]:

THEOREM 2. *Let $X=[t]$, where $0\leqslant t\leqslant n$. Let $\mathbf{B}=\bigcup_{m=t}^{n}\mathbf{B}_m(X)$. Then $\bar{v}(\mathbf{B})=2^{n-t}$. Moreover, there exists a partition of $2^{[n]}$ into just 2^{n-t} anticlusters for $\mathbf{B}$ using the ∇ operation.*

5. The Greedy Construction

We cannot see how to explicitly construct the matrix M for $\mathbf{B}_n(X)$ for general X. An approach that works in many cases, but not in general, is to select the columns one at a time, beginning at column 1. In this greedy approach, we select any column vector for column j that is, for every i such that $j\in(X+i)(\bmod n)$, independent of the subspace generated by the columns indexed by the set $(X+i)$ $(\bmod n)\cap[j-1]$. This means simply that we do not choose anything for column j, given columns $1, 2, ..., j-1$, that disqualifies M.

Our main new result using this method is to show that the analogs of Conjectures 1, 2, and 3 hold for the ordinary (not cyclic) translates of X, $\mathbf{B}_n^*(X)$. Previously it was only known that $\lim_{n\to\infty} v(\mathbf{B}_n^*(X))/2^n$ exists [1], but the value of the limit was not established.

THEOREM 3. *For all t and n with $0\leqslant t\leqslant n$, and all t-subsets X of $[n]$, $v(\mathbf{B}_n^*(X))=\bar{v}(\mathbf{B}_n^*(X))=2^{n-t}$.*

Proof. We assume $t>0$ since $t=0$ is trivial. Each element j of $[n]$ appears at most once as the kth largest element of a translate of X, for each k with $1\leqslant k\leqslant t$. To select column j given the first $j-1$ columns of M, we must only be sure that for each such k, column j is not an element of the subspace generated by the columns indexed by the $k-1$ elements less that j in the translate of X. This space of dimension $k-1$ has $2^{k-1}-1$ nonzero elements. In addition, column j must be nonzero itself. So the number of columns which will work as column j is at least

$$2^t-\left(1+\sum_{k=1}^{t}(2^{k-1}-1)\right)=t>0.$$

So it is always possible to select column j and, therefore, to construct M. The theorem follows by the arguments in Sections 2 and 3. ∎

For cyclic translates of a consecutive block, $\mathbf{B}_n([t])$, we already constructed a matrix M in the last section. From this we deduced Theorems 1 and 2. We now show that the greedy approach is always successful in producing M for $\mathbf{B}_n([t])$.

THEOREM 4. *The greedy approach always constructs a suitable matrix for* $\mathbf{B}_n([t])$.

Proof. The first t columns can be any set of independent columns. For convenience of argument, assume they form an identity matrix. It must be proven that for each $a=1, 2, ..., n-t$ there exists a satisfactory choice for column $t+a$ given satisfactory columns 1 through $t+a-1$. It is sufficient (but stronger than necessary) to prove for each a that there exists a choice of column $t+a$ such that for all b with $0\leqslant b\leqslant t-1$, this column is independent of the union of the preceding b columns already selected and the initial $t-1-b$ columns. Since these initial columns come from the identity, it suffices to prove the following fact. For a vector $\mathbf{v}$ in $\mathbf{Z}_2^t$, let $\mathbf{v}(a)$ denote the vector in $\mathbf{Z}_2^a$ consisting of the bottom a components of $\mathbf{v}$.

LEMMA. *For any $t-1$ vectors $\mathbf{v}_1, ..., \mathbf{v}_{t-1}$ in $\mathbf{Z}_2^t$, there exists $\mathbf{v}_t$ in $\mathbf{Z}_2^t$ such that for all b with $0\leqslant b\leqslant t-1$, $\mathbf{v}_t(b+1)$ is linearly independent of* $\{\mathbf{v}_{t-b}(b+1), \mathbf{v}_{t-b+1}(b+1), ..., \mathbf{v}_{t-1}(b+1)\}$.

Proof of Lemma. The lemma holds for $t=1$ by setting $\mathbf{v}_t=(1)$. Assume that it holds for $t-1$. Let $\mathbf{v}_1, ..., \mathbf{v}_{t-1}\in\mathbf{Z}_2^t$. First suppose that the $t-1$ vectors $\mathbf{v}_1(t-1), ..., \mathbf{v}_{t-1}(t-1)$ are independent. By induction there exists $\mathbf{w}$ in $\mathbf{Z}_2^{t-1}$ such that $\mathbf{w}(b+1)$ is independent of $\mathbf{v}_{t-b}(b+1), ..., \mathbf{v}_{t-1}(b+1)$ for all $0\leqslant b\leqslant t-2$. Since $\mathbf{v}_1(t-1), ..., \mathbf{v}_{t-1}(t-1)$ form a basis for $\mathbf{Z}_2^{t-1}$, $\mathbf{w}$ can be expressed uniquely as a sum of some subset of these, say $\mathbf{w}=\sum_{i\in I}\mathbf{v}_i(t-1)$, where $I\subseteq[t-1]$ is unique. To form a suitable $\mathbf{v}_t$ it suffices to top off $\mathbf{w}$ by a component not equal to that of $\sum_{i\in I}\mathbf{v}_i$, so take $\mathbf{v}_t=\sum_{i\in I}\mathbf{v}_i+\mathbf{e}_1$, where $\mathbf{e}_1=(1, 0, ..., 0)^{\mathrm{T}}$.

Now consider the case in which $\mathbf{v}_1(t-1), ..., \mathbf{v}_{t-1}(t-1)$ are dependent. Choose j as small as possible so that $\mathbf{v}_j(t-1)$ belongs to the span of $\mathbf{v}_{j+1}(t-1), ..., \mathbf{v}_{t-1}(t-1)$. By inductive hypothesis, there exists a vector $\mathbf{w}$ in $\mathbf{Z}_2^{t-1}$ that is suitable for $\mathbf{v}_1(t-1), ..., \mathbf{v}_{j-1}(t-1), \mathbf{v}_{j+1}(t-1), ...,$

$\mathbf{v}_{t-1}(t-1)$. For $0 \leqslant b \leqslant t-2$, $\mathbf{w}(b+1)$ is independent of (i.e., not in the span of) the bottom $b+1$ components of the last b vectors in this list. It follows that $\mathbf{w}$ is suitable for $\mathbf{v}_2(t-1), ..., \mathbf{v}_{t-1}(t-1)$, and that $\mathbf{w}$ is independent of $\mathbf{v}_1(t-1), ..., \mathbf{v}_{t-1}(t-1)$. Hence, either of the two vectors $\mathbf{v}$ in $\mathbf{Z}_2^t$ with $\mathbf{v}(t-1) = \mathbf{w}$ is a suitable choice for $\mathbf{v}_t$. ∎

By the lemma, the greedy selection never gets stuck, so the theorem is proven. ∎

Suppose that we first pick the t columns of the identity matrix and that thereafter we impose the condition of the proof of the lemma on each selected column instead of the weaker greedy algorithm requirement. By induction using the counting argument in the proof of the lemma, there is precisely one choice for each new column. The unique $t \times n$ matrix M for $\mathbf{B}_n([t])$ that this produces must therefore be the matrix described in the proof of Theorem 1.

6. Results for $\mathbf{B}_n(X)$ for General X

Although the greedy construction always works for the cyclic translates $\mathbf{B}_n([t])$ of a connected block $[t]$, it unfortunately does not work for $\mathbf{B}_n(X)$ for general X.

Example. Let $X = \{1, 2, 4\}$ and $n = 8$. The following matrix cannot be extended:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & * \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & * \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & * \end{bmatrix}.$$

For all i, the columns indexed by $(X+i) \cap [7]$ are linearly independent, yet there is no choice for the last column so that the matrix satisfies Conjecture 3. A matrix that does work is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

In contrast to the negative result above, it has been shown in [1] that

$$\lim_{n \to \infty} v(\mathbf{B}_n(X))/2^n$$

exists for all X, call it $r(X)$, and that

$$r(X)=\lim_{n\to\infty} v(\mathbf{B}_n^*(X))/2^n.$$

Considering this along with Theorem 3 gives us the following result.

THEOREM 5. *For all X, $r(X)=2^{-t}$.*

The cited result in [1], upon which Theorem 5 depends, was proven using known facts about the entropy function from information theory. A more direct proof of Theorem 5 has eluded us. However, we will apply our result for non-cyclic translates of X, Theorem 3, to prove that $v(\mathbf{B}_n(X))/2^n$ exactly equals 2^{-t} for infinitely many values of n. Along with Theorem 5, this offers strong evidence in support of Conjectures 1–3. We denote $\mathbf{Z}^+=\{1, 2, ...\}$.

THEOREM 6. *For every z in $\mathbf{Z}^+$ there exists m depending on z such that for all multiples n of m and all $X\subseteq[z]$, $v(\mathbf{B}_n(X))=2^{n-|X|}$.*

Proof. Let $t\in[z]$ and let $X\in\binom{[z]}{t}$. Let x denote the largest element in X. Let $n\in\mathbf{Z}^+$, $n\geqslant x$. Apply the greedy construction to form a $t\times n$ matrix M_n, suitable for $\mathbf{B}_n^*(X)$, which exists by Theorem 3. Since there are only finitely many $t\times(x-1)$ (0, 1)-matrices, it follows that for sufficiently large n there exist two disjoint connected blocks of $x-1$ columns in M_n that are identical. Let a (resp. b) be the index of the first column in the first (resp. second) block. Let $d=d(X)=b-a$ and let N be the $t\times d$ matrix consisting of columns $a, a+1, ..., b-1$ of M_n. This matrix N is suitable for $\mathbf{B}_d(X)$, not merely $\mathbf{B}_d^*(X)$. For r in $\mathbf{Z}^+$, the matrix formed by taking r copies of N side-by-side is suitable for $\mathbf{B}_{rd}(X)$. Hence $v(\mathbf{B}_n(X))=2^{n-t}$ for all multiples n of d. Further, d can be bounded above in terms of z alone, independent of $X\subseteq[z]$, say $d(X)\leqslant d_z$. Let m be divisible by each element of $[d_z]$, e.g., $m=d_z!$. Then every multiple of m is divisible by $d(X)$ for all $X\subseteq[z]$, and the theorem follows. ∎

7. COMPUTATIONAL EVIDENCE FOR CONJECTURE 3

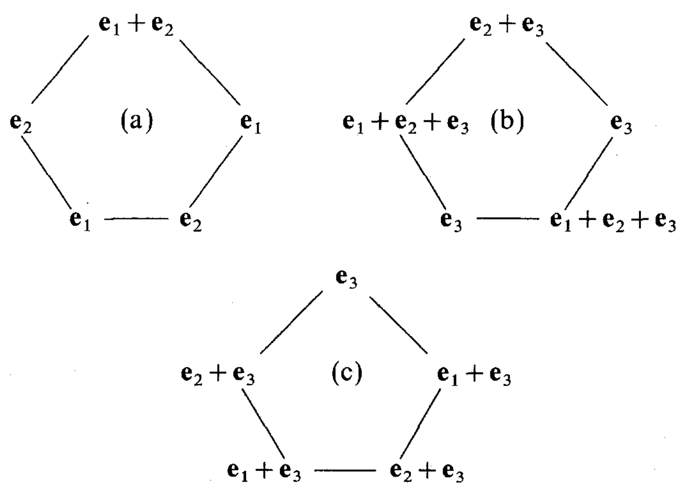
First we point out two rather obvious reductions that apply for all t. Let $X=\{x_1<x_2<\cdots<x_t\}$, and let $a_1, a_2, ..., a_t$ denote the cyclic distances among $x_1, x_2, ..., x_t$. That is, $a_1=x_2-x_1, ..., a_{t-1}=x_t-x_{t-1}$, $a_t=n+x_1-x_t$, so $a_1+\cdots+a_t=n$.

(1) Suppose that $a_1, ..., a_t$ have a common factor $k>1$. Then the problem decomposes into k disjoint problems in which $n, a_1, ..., a_t$ have all been divided by k.

(2) Suppose that some a_i is relatively prime to n. If we relabel the n-cycle by walking around it in steps of size a_i, we obtain a new problem in which a_i has been replaced by 1.

Now let us restrict our attention to the special case $t = 3$.

(3) Suppose that one of a_1, a_2, a_3, say a_1, has a common factor with n, but that reduction (1) *cannot* be applied. In this case Conjecture 3 is true. In the interest of brevity, we will only sketch the proof. Suppose that k is the greatest common divisor of a_1 and n. In fact k must be relatively prime to both a_2 and a_3; otherwise the equation $a_1 + a_2 + a_3 = n$ would allow us to show that reduction (1) applies. Now let us decompose the n-cycle into k cycles of length n/k, by walking around it in steps of size a_1. These minor cycles are not independent; for each cyclic translate of X, two elements are in one minor cycle and the third element is in another minor cycle. As a result of the fact that a_2 and a_3 relatively prime to k, these dependencies arrange the minor cycles in a major cycle of length k. The most complicated case is when both major and minor cycles are of odd length. We use three possible assignments of vectors to minor cycles, illustrated below. Note that in each case the vectors alternate except at one position. Similarly, in the major cycle, we alternate types (a) and (b), except that we use (c) at one place. In the dependencies from a type (b) cycle to a type (a) cycle, or a type (c) cycle to a type (b) cycle, the relative orientations of the minor cycles are constrained. However, the dependencies from type (a) to the other types are not constrained, and this gives us enough flexibility to construct the major cycle.



By combining (1) and (3), we may assume that n is odd. By combining (2) and (3), we may assume that $a_1 = 1$. Using these reductions, we have checked Conjecture 3 for $t = 3$ and all $n \leqslant 200$ by computer search.

8. A Conjecture for Subspaces of $\mathbf{Z}_2^n$

Thus far we have been considering how to build a matrix M that is suitable for given $\mathbf{B} \subseteq \binom{[n]}{t}$ by viewing the columns as vectors in $\mathbf{Z}_2^t$. Now we try looking at what happens when we view the rows of M as vectors in $\mathbf{Z}_2^n$.

In $\mathbf{Z}_2^n$, $\mathbf{0}$ denotes the zero vector. For j in $[n]$, $\mathbf{e}_j$ denotes the jth standard basis vector. For $\mathbf{v}$, $\mathbf{w}$ in $\mathbf{Z}_2^n$, $\mathbf{v} \cdot \mathbf{w}$ denotes the usual inner product, $\sum v_i w_i$ (mod 2). If $\mathbf{T}$ is a subspace of $\mathbf{Z}_2^n$, then

$$\mathbf{T}^\perp = \{\mathbf{v} \in \mathbf{Z}_2^n \colon \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all} \quad \mathbf{w} \in \mathbf{T}\}.$$

So $\mathbf{T}^\perp$ is a subspace of dimension $n - \dim(\mathbf{T})$. Note that $\mathbf{T}$ and $\mathbf{T}^\perp$ are not disjoint in general. For $\mathbf{B} \subseteq [n]$, let $\mathbf{S}(B)$ denote the subspace generated by the vectors $\mathbf{e}_j$, for $j \in B$. The projection operator $\pi_{\mathbf{S}(B)} \colon \mathbf{Z}_2^n \to \mathbf{S}(B)$ acts on $\mathbf{v} \in \mathbf{Z}_2^n$ by setting v_j to 0 for all $j \notin B$.

Recall that we say a $t \times n$ matrix M is *suitable for* $\mathbf{B}$ if $\mathbf{B} \subseteq \binom{[n]}{t}$ and for all B in $\mathbf{B}$ the columns of M indexed by B are linearly independent. Considering the rows of M as a basis for a t-dimensional subspace $\mathbf{T}$ of $\mathbf{Z}_2^n$ and vice versa, we observe the following:

Proposition 1. *For $\mathbf{B} \subseteq \binom{[n]}{t}$ there exists a suitable matrix M if and only if there exists a t-dimensional subspace $\mathbf{T}$ of $\mathbf{Z}_2^n$ such that for all B in $\mathbf{B}$, $\pi_{\mathbf{S}(B)}(\mathbf{T}) = \mathbf{S}(B)$.*

We are ready to state the main result of the section.

Theorem 7. *For $\mathbf{B} \subseteq \binom{[n]}{t}$ there exists a suitable matrix M if and only if there exists a subspace $\mathbf{U}$ of $\mathbf{Z}_2^n$ of dimension $n - t$ such that for all B in $\mathbf{B}$, $\mathbf{U} \cap \mathbf{S}(B) = \{\mathbf{0}\}$.*

Proof. In light of the proposition, it suffices to show that for each t-dimensional subspace $\mathbf{T}$ of $\mathbf{Z}_2^n$ and each B in $\binom{[n]}{t}$,

$$\pi_{\mathbf{S}(B)}(\mathbf{T}) = \mathbf{S}(B) \qquad \text{if and only if} \quad \mathbf{T}^\perp \cap \mathbf{S}(B) = \{\mathbf{0}\}. \tag{5}$$

To show this, let us first show that

$$\pi_{\mathbf{S}(B)}(\mathbf{T}) = \mathbf{S}(B) \qquad \text{if and only if} \quad \mathbf{T} \cap \mathbf{S}(B)^\perp = \{\mathbf{0}\}. \tag{6}$$

Since $\mathbf{T}$ and $\mathbf{S}(B)$ have the same dimension, the map $\pi_{\mathbf{S}(B)} \colon \mathbf{T} \to \mathbf{S}(B)$ is surjective if and only if it is injective. Therefore (6) follows from the fact that $\ker \pi_{\mathbf{S}(B)} = \mathbf{T} \cap \mathbf{S}(B)^\perp$.

Next, we show that

$$\mathbf{T} \cap \mathbf{S}(B)^\perp = \{\mathbf{0}\} \qquad \text{if and only if} \quad \mathbf{T}^\perp \cap \mathbf{S}(B) = \{\mathbf{0}\}. \tag{7}$$

To see that, note that if $\mathbf{T} \cap \mathbf{S}(B)^{\perp} = \{\mathbf{0}\}$, then $\mathbf{Z}_2^n = (\mathbf{T} \cap \mathbf{S}(B)^{\perp})^{\perp} = \mathbf{T}^{\perp} + \mathbf{S}(B)$, hence for dimensional reasons $\mathbf{T}^{\perp} \cap \mathbf{S}(B) = \{\mathbf{0}\}$. Now (5) follows from (6) and (7). ∎

Applying the theorem to $\mathbf{B} = \mathbf{B}_n(X)$ yields an equivalent version of Conjecture 3.

CONJECTURE 3′. *For all t and n with $0 \leqslant t \leqslant n$, and all X in $\binom{[n]}{t}$, there exists an $(n-t)$-dimensional subspace $\mathbf{U}$ of $\mathbf{Z}_2^n$ such that $\mathbf{U} \cap \mathbf{S}(X+i) = \{\mathbf{0}\}$ for all i.*

One more interesting observation follows from the proposition and the proof of Theorem 7:

THEOREM 8. *For $\mathbf{B} \subseteq \binom{[n]}{t}$ there exists a suitable matrix M if and only if there exists a suitable matrix for $\bar{\mathbf{B}} = \{[n]-B \colon B \in \mathbf{B}\} \subseteq \binom{[n]}{n-t}$.*

Proof. Replacing $\mathbf{T}$ and $\mathbf{S}(B)$ by $\mathbf{T}^{\perp}$ and $\mathbf{S}([n]-B)$ in statement (6) of the proof of Theorem 7, we obtain

$$\pi_{\mathbf{S}([n]-B)}(\mathbf{T}^{\perp}) = \mathbf{S}([n]-B) \qquad \text{if and only if } \mathbf{T}^{\perp} \cap \mathbf{S}([n]-B)^{\perp} = \{\mathbf{0}\}. \quad (8)$$

Combining (8), (5), and the fact that $\mathbf{S}([n]-B)^{\perp} = \mathbf{S}(B)$ produces

$$\pi_{\mathbf{S}([n]-B)}(\mathbf{T}^{\perp}) = \mathbf{S}([n]-B) \qquad \text{if and only if} \quad \pi_{\mathbf{S}(B)}(\mathbf{T}) = \mathbf{S}(B).$$

This, together with the proposition, proves the result. ∎

Thus to prove Conjectures 3 and 3′ it suffices to assume that $t \leqslant \frac{1}{2}n$ (or to assume $t \geqslant \frac{1}{2}n$).

9. CONCLUDING REMARKS

For X in $\binom{[n]}{t}$ the three families of subsets $\{X\}$, $\mathbf{B}_n(X)$, and $\binom{[n]}{t}$ are all generated by the action of a subgroup of the symmetric group S_n on the set X (by the trivial subgroup, by the cyclic subgroup of order n, and by the full group S_n, respectively). So far we can find no connection between the conjectures about $\mathbf{B}_n(X)$ and the cyclic group. Therefore we put forth this broader conjecture for further study:

CONJECTURE 4. *Let $\mathbf{B}$ be a nonempty subset of $\binom{[n]}{t}$ such that each member i of $[n]$ belongs to at most t members of $\mathbf{B}$. Then $v(\mathbf{B}) = 2^{n-t}$.*

Note added in proof. Z. Füredi, J. R. Griggs, and D. J. Kleitman have obtained a proof of the stronger matrix version of Conjecture 4 for the case $t=3$. This will appear soon in this journal.

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